The general crossing relation for boundary reflection matrix

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Abstract

In this paper, we give the general crossing relation for boundary reflection matrix $R(\beta)$,which is the extension of the work given by Ghoshal and Zamolodchikov .We also use the first non-trivial extended crossing relation to determine the scaler factor of $R(\beta)$ which is the rational diagonal solution to the boundary Yang-Baxter equation in the case of l=2 and n=3.

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1 Introduction

Much work has recently been done in integrable quantum field theory and lattice statistical mechanics on models with a boundary , where the integrability is guaranteed by the boundary Yang-Baxter equation (BYBE) . An exact solution to such a field theory could provide a better undstanding of boundary-related phenomena in statistical systems near criticality[1]. In statistical mechanics,the emphasis has been on deriving solutions of BYBE and calculation of various surface critical phenomena, both at and away from criticality[8,9].

An important character of an integrable field theory is its factorizable scattering matrix (S-matrix). In the "bulk theory" , the factorizable S-matrix is completely determined in terms of the two-particle S-matrix ,which should satisfy the Yang-Baxter equation(YBE), in addtion to the standard equations of unitarity and crossing relation[2,3] . The general crossing relation was given by Y.H. Quano[6] .These equations have much restrictive power, determining the S-matrix up to the so-called "CDD ambiguity" .Recently, much important progress in the boundary integrable model is that Ghoshal and Zamolodchikov gave the approprite anolog of the crossing relation for boundary reflection matrix (R-matrix) in the Ref.[3].Together with BYBE and unitarity for boundary R-matrix,the above equations have exactly the same restrictive power as the corresponding "bulk " system ,i.e. they allow one to pin down the facterizable boundary R-matrix up to the "CDD factors" . This crossing relation also play a very important role in solving boundary integrable spin chain (e.g boundary XXZ model [4] and boudary XYZ model[5]). However, it was known that the crossing relation gave by Ghoshal and Zamolodchikov is only the special case (n=2) of the general one. In this paper, we will give the general version of crossing relation for boundary R-matrix (i.e for any $A_n^{(1)}$ case)

2 The general boundary crossing relation

For an integrable boundary massive field theory, the scattering theory is purely elastic and the corresponding scattering matrix is factorizable. Besides the bulk scattering processes (which is described by two-particle S-matrix $S(\beta)$),there exist a process which represents the particles reflecting with the boundary.This process is described by boundary R-matrix $R(\beta)$ .The factorization of scattering in the boundary case is equivalent to validity of YBE and BYBE

$$\begin{aligned} \text{YBE} \quad : \quad & S_{12}(\beta_1-\beta_2)S_{13}(\beta_1-\beta_3)S_{23}(\beta_2-\beta_3) \\ & = S_{23}(\beta_2-\beta_3)S_{13}(\beta_1-\beta_3)S_{12}(\beta_1-\beta_2) \quad , \qquad (1) \\ \text{BYBE} \quad : \quad & S_{12}(\beta_1-\beta_2)R_1(\beta_1)S_{21}(\beta_1+\beta_2)R_2(\beta_2) \\ & = R_2(\beta_2)S_{12}(\beta_1+\beta_2)R_1(\beta_1)S_{21}(\beta_1-\beta_2) \quad , \qquad (2) \end{aligned}$$

where $S(\beta)$ is the same one as in the bulk case and boundary R-matrix should satisfy the requirements of unitary

$$R(\beta)R(-\beta) = 1 \quad . \qquad (3)$$

AS the usual conventions,$S_{mn}(\beta)$ signifies the matrix on $V \otimes V ... \otimes V$ acting on the m-th and n-th component and as identity on the other ones, and $R_k(\beta)$ signifies the matrix on tensor space acting on th k-th component and as identity on the other ones.

2.1 The general boundary crossing relation

We only consider the solution to YBE which has the following property

$$S(-w) = P_2^{(-)} \times (\text{something}) \quad , \qquad (4)$$

where w is so-called crossing parameter (it is usally rescaling as $i\pi$) ,and $P_2^{(-)}$ is the completely antisymmetric project operator in $V^{\otimes 2}$. In fact, in addition to rational and elliptic solution to YBE having properties (4) ,there exist a series trigonometric solution to YBE (which is the trigonometrical limit of Zn Belavin' solution) carrying with property (4).

Due to the property (4),one can construct the completely antisymmetric fusion procedure for R-matrix[10,11,12].First, we give some notation about fusion. Let $V = C^n$ (2$\leq$ n) and V^* be the dual space of V .Then we have a homorphism C :

$$C: \quad V^* \longrightarrow \Lambda^{n-1}(V) \quad \text{and}$$
$$Ce_i^* = \frac{1}{\sqrt{(n-1)!}} \epsilon_i^{i_1 \ldots i_{n-1}} e_{i_1} \otimes \ldots \otimes e_{i_{n-1}} \qquad ,$$

where e_i^* is the base of dual space V^* and $\epsilon_i^{i_1,\ldots,i_{n-1}}$ is the n-th order completely antisymmetric tensor. Through the fusion procedure for R-matrix ,one can obtain

Theorem 2.1 For any integral l $(l \leq n)$, define

$$R^{(l)}(\beta): \qquad V^{\otimes l} \longrightarrow V^{\otimes l} \quad \text{by}$$
$$R^{(l)}(\beta) = R_l(\beta + lw) S_{l-1,l}(2\beta + (2l-1)w) \ldots R_1(\beta + w)$$

then image of $R^{(l)}(\beta)|_{\Lambda^{(l)}(V)}$ is also in $\Lambda^{(l)}(V)$ .Furthermore,we can define

$$R^*(\beta): \qquad V^* \longrightarrow V^* \quad \text{by}$$

$$\begin{aligned} R^*(\beta) &= C^{-1} P_{n-1}^{(-)} R_n(\beta + (n-1)w) S_{n-1,n}(2\beta + (2n-3)w) \\ &\ldots.. R_2(\beta + w) P_{n-1}^{(-)} C \qquad . \end{aligned} \tag{5}$$

[**Proof :**]

From YBE and the properties (4), it is easy to see the following properties

$$\begin{aligned} & S_{12}(-w) S_{13}(-2w) \ldots S_{1l}(-(l-1)w) S_{23}(-w) \ldots. S_{l-1,l}(-w) \\ &\quad = P_l^{(-)} \times M \qquad , \end{aligned} \tag{6}$$

$$\begin{aligned} & S_{l,l-1}(-w) S_{l,l-2}(-2w) \ldots S_{l1}(-(l-1)w) S_{l-1,l-2}(-w) \ldots. S_{21}(-w) \\ &\quad = P_l^{(-)} \times M' \qquad , \end{aligned} \tag{7}$$

where $P_l^{(-)}$ is completely antisymmetric project operator in $V^{\otimes l}$ and $M, M' \in \mathrm{End}(V^{\otimes l})$ .Using the YBE and BYBE, one have

$$\begin{aligned}
&[S_{12}(-w)S_{13}(-2w)...S_{1l}(-(l-1)w)S_{23}(-w)....S_{l-1,l}(-w)] \times \\
&\quad R_1(\beta+w)S_{21}(2\beta+3w)...S_{l1}(2\beta+(l+1)w)R_2(\beta+2w) \\
&\quad ...R_l(\beta+lw) \\
&= R^{(l)}(\beta)[S_{l,l-1}(-w)S_{l,l-2}(-2w)...S_{l1}(-(l-1)w)S_{l-1,l-2}(-w) \\
&\quad ....S_{21}(-w)] \qquad . \qquad (8)
\end{aligned}$$

Thanks to properties (6) and (7), this means that

$$P_l^{(-)}R^{(l)}(\beta)P_l^{(-)} = R^{(l)}(\beta)P_l^{(-)} \quad , \qquad (9)$$

so it ensure us that $\mathrm{Im}(R^{(l)}(\beta))|_{\Lambda^{(l)}(V)} \in \Lambda^{(l)}(V)$.
We are now in a position to mention the crossing relation for generic n .Because the crossing relation of boundary R-matrix is mainly considered,let us assume that our $R(\beta)$ has already enjoyed in the unitarity :$R(\beta)R(-\beta) = 1$, and $S(\beta)$ enjoyed in unitarity and crossing symmetry[6]

$$\begin{aligned}
&S_{12}(\beta)S_{21}(-\beta) = 1 \quad , \\
&[S^{V,V^*}(\beta)]_{ik^*}^{jl^*} = S(-\beta-nw)_{li}^{kj} \quad , \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
S^{V,V^*}(\beta) &= (1\otimes C^{-1})(1\otimes P_{n-1}^{(-)})S_{1n}(\beta+(n-1)w)....S_{12}(\beta+w) \\
&\quad (1\otimes P_{n-1}^{(-)})(1\otimes C) \qquad .
\end{aligned}$$

Using the definition of $R^*(\beta)$,the unitarity of $S(\beta)$ and $R(\beta)$,one can find $R^*(\beta)$ with the following inversion relation

$$R^*(\beta)R^*(-\beta-nw) = 1 \quad . \qquad (11)$$

Theorem 2.2 $R^*(\beta)$ which is obtained through fusion of $R(\beta)$ can also be directly expressed into

$$[R^*(\beta)]_j^k = D(\beta)[S^{V,V}(2\beta+nw)]_{kl'}^{lj}[R(-\beta-nw)]_l^{l'} \quad , \qquad (12)$$

where $D(\beta)$ is a scaler factor (actually it is the quantum determinator of $R(\beta)$)

[**Proof:**]

Owing to the fact that dim($\Lambda^n(V)$ is equal to 1 and Theorem 2.1 ,one have

$$\sum_{m,l,l'} [R^*(\beta)]^k_m [S^{V,V^*}(2\beta)]^{jm}_{l'l} [R(\beta)]^{l'}_l = D(\beta)\delta^{jk} \quad ,$$

where $D(\beta)$ is a scaler factor which depends on $R(\beta)$ and $S(\beta)$ .Using the inversion relation (11) and the crossing symmetry of $S(\beta)$ (10), one can obtain

$$[R^*(\beta)]^k_j = D(\beta)[S^{V,V}(2\beta + nw)]^{lj}_{kl'}[R(-\beta - nw)]^{l'}_l \quad . \qquad (12a)$$

Due to (12),one can reduce $D(\beta)$ to 1 through rescaling the boundary matrix $R(\beta)$ .Therefore,one can determine the scaler factor of $R(\beta)$ using the following relation

$$[R^*(\beta)]^k_j = [S^{V,V}(2\beta + nw)]^{lj}_{kl'}[R(-\beta - nw)]^{l'}_l \quad .$$

This relation can be considered as the generalization crossing relation for generic n.When one considers V^* ,circumstances slightly change between n=2 and $n > 2$ case . Speaking in terms of Young tableau , $V = \square$,the fundamental representation sl_n, while V^* corresponds to vertical n-1 $\square$'s ,the space of antisymmetric tensors in $V^{\otimes n-1}$ .Thus V and V^* can be identified even at the level of Young tableau if and only if n=2 .For $n > 2$, more complex procedure than (5) is needed and (12a) is non-trivial extension of crossing relation for $n > 2$. However,when n is equal to 2 ,the restricted condition for scaler factor of $R(\beta)$ become the same as that given by Ghoshal and Zamolodchikov.We have checked it for the rational ,trigonometric and elliptical type solution to BYBE.

Hence,in order to determine the exact boundary reflecting matrix ,one can impose the following condition on $R(\beta)$ (BYBE,unitarity,crossing relation)

$$\begin{aligned} \text{BYBE} \quad : \quad & S_{12}(\beta_1 - \beta_2)R_1(\beta_1)S_{21}(\beta_1 + \beta_2)R_2(\beta_2) \\ & = R_2(\beta_2)S_{12}(\beta_1 + \beta_2)R_1(\beta_1)S_{21}(\beta_1 - \beta_2) \quad , \qquad (13) \end{aligned}$$

$$\text{unitarity} \qquad R(\beta)R(-\beta) = 1 \quad , \tag{14}$$

$$\text{crossing relation} \quad : \ [R^*(\beta)]_j^k = [S^{V,V}(2\beta + nw)]_{kl'}^{lj} [R(-\beta - nw)]_l^{l'} \tag{15}$$

The above condition can only reduce the scaler factor up to some "CDD" factor $\Phi(\beta)$ which should also be generalized as satisfying follow requirement

$$\Phi(\beta)\Phi(-\beta) = 1 \ , \ \prod_{l=1}^{n} \Phi(\beta + lw) = 1 \quad . \tag{16}$$

When n=2 ,the above requirement return the usual one.Moreover,this ambiguity can be canceled by some dynamical requirement[2,3].

2.2 An example

To determine the exact boundary reflecting matrix $R(\beta)$,the following procedure are usually taken : first,one obtain the solution to (13) upon to some scaler factor;then, the unitarity (14) and crossing relation (15) are used to determine the scaler factor.In this subsection, we obtain the rational diagonal solution to BYBE for generic n. Then without losing the generality, the first nontrivial extension of crossing relation are applied to determine the scaler factor for n=3.

Here,we consider the rational solution to YBE with unitarity and crossing relation satisfied,which is given as follows

$$\begin{aligned} S(\beta) &= a(\beta)1 + b(\beta)P \\ &= k(\beta)(\frac{\beta}{\beta + w}1 + \frac{w}{\beta + w}P) \quad , \end{aligned}$$

where P is the permutation operator in $V^{\otimes 2}$:

$$P(e_i \otimes e_j) = e_j \otimes e_i$$

and scaler factor $k(\beta)$ is equal to

$$k(\beta) = -\frac{\Gamma(\frac{\beta}{nw})\Gamma(\frac{-\beta}{nw} + \frac{1}{nw})}{\Gamma(\frac{-\beta}{nw})\Gamma(\frac{\beta}{nw} + \frac{1}{nw})} \quad .$$

The BYBE leads to the following functional equations

$$
\begin{aligned}
&a(\beta_1+\beta_2)b(\beta_1-\beta_2)R_j^k(\beta_1)R_i^l(\beta_2)+a(\beta_1-\beta_2)b(\beta_1+\beta_2)R_i^{i'}(\beta_1)R_{i'}^l(\beta_2)\delta_j^k\\
&\qquad +b(\beta_1+\beta_2)b(\beta_1-\beta_2)R_j^{i'}(\beta_1)R_{i'}^l(\beta_2)\delta_i^k\\
&=a(\beta_1+\beta_2)b(\beta_1-\beta_2)R_i^l(\beta_1)R_j^k(\beta_2)+a(\beta_1-\beta_2)b(\beta_1+\beta_2)R_{i'}^k(\beta_1)R_j^{i'}(\beta_2)\delta_i^l\\
&\qquad +b(\beta_1+\beta_2)b(\beta_1-\beta_2)R_{i'}^l(\beta_1)R_j^{i'}(\beta_2)\delta_i^k \qquad .
\end{aligned}
$$

In the diagonal case ,i.e $R_i^j(\beta)=R_i(\beta)\delta_i^j$, the above equations become

$$
\begin{aligned}
&a(\beta_1+\beta_2)b(\beta_1-\beta_2)[R_j(\beta_1)R_i(\beta_2)-R_i(\beta_1)R_j(\beta_2)]\\
&=a(\beta_1-\beta_2)b(\beta_1+\beta_2)[R_j(\beta_1)R_j(\beta_2)-R_i(\beta_1)R_i(\beta_2)] \quad .
\end{aligned} \tag{17}
$$

The solution to (17) is

$$
R(\beta)=\begin{cases} r(\beta) \quad , & if \quad i\le n-l \\ r(\beta)\frac{\xi-\beta}{\xi+\beta} \quad , & otherwise \end{cases} \tag{17a}
$$

This diagonal solution has two parameters :$l\in N$ and $\xi\in C$,which has the same structure as the diagonal solution for trigonometric solution given by de Vega[7] .

Only for simplicity but without losing the generality,we only consider the crossing relation of this solution (17a) for l=2 and n=3,namely,$R(\beta)$ is equal to

$$
r(\beta)\begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{\xi-\beta}{\xi+\beta} \end{pmatrix} \quad . \tag{18}
$$

Using the fusion for $R(\beta)$,one can get

$$
\begin{aligned}
R^*(\beta)&=\frac{r(\beta+w)r(\beta+2w)k(2\beta+3w)(\xi-\beta-2w)(\beta+w)}{(\beta+2w)(\xi+\beta+w)}\\
&\times\begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{\xi+\beta+w}{\xi-\beta-2w} \end{pmatrix} \quad .
\end{aligned} \tag{19}
$$

On the other hand,from the crossing relation (15), one should obtain

$$
\begin{aligned}
R^*(\beta)&=\frac{r(-\beta-3w)k(2\beta+3w)(\beta+3w)(\xi-\beta-2w)}{(\beta+2w)(\xi-\beta-3w)}\\
&\times\begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{\xi+\beta+w}{\xi-\beta-2w} \end{pmatrix} \quad .
\end{aligned} \tag{20}
$$

So, unitarity and crossing relation (14) and (15) lead to the requirements for the scaler factor $r(\beta)$

$$r(\beta)r(-\beta)=1 \quad ,$$
$$r(\beta+w)r(\beta+2w)r(\beta+3w)=\frac{(\xi+\beta+w)(\beta+3w)}{(\xi-\beta-3w)(\beta+w)} \quad .$$

Solving the above equations,one can obtain the scaler factor $r(\beta)$ upon to "CDD" ambiguity,

$$\begin{aligned} r(\beta) = & -\frac{\Gamma(\frac{\beta}{3w})\Gamma(\frac{\xi+\beta}{3w}+\frac{1}{3})}{\Gamma(-\frac{\beta}{3w})\Gamma(\frac{\xi-\beta}{3w}+\frac{1}{3})} \\ & \times\frac{\Gamma(-\frac{\beta}{3w}+\frac{1}{3})\Gamma(\frac{\xi-\beta}{3w})}{\Gamma(\frac{\beta}{3w}+\frac{1}{3})\Gamma(\frac{\xi+\beta}{3w})}\times\Phi(\beta) \quad , \end{aligned} \tag{21}$$

where $\Phi(\beta)$ is a "CDD" factor which satisfies equations

$$\begin{aligned} & \Phi(\beta)\Phi(-\beta)=1 \quad , \\ & \Phi(\beta+w)\Phi(\beta+2w)\Phi(\beta+3w)=1 \quad . \end{aligned} \tag{22}$$

The equations (22) which "CDD" factors should be satisfied is the generalized version of n=2 case .

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References

[1] K.Binder,in Phase Transitions and Critical Phenomena,Vol.8, eds. C.Domb and J.Lebowitz (Academic ,London,1983).

[2] A.B.Zamolodchikov and Al.B.Zamolodchikov, Annals of Phys.120 (1979),253.

[3] S.Ghoshal and Al.B.Zamolodchikov, Int.J.Mod.Phys. A21 (1994) 3841.

[4] M.Jimbo ,R.Kedem, T.Kojima, H.Konno and T.Miwa, Nucl.Phys. B441 (1995)437.

[5] M.Jimbo ,R.Kedem, H.Konno ,T.Miwa and R.Weston,Nucl.Phys. B448 (1995)429.

[6] Y.H.Quano ,Mod.Phys.Lett.A Vol.8 No.35 (1993) 3363.

[7] H.J. de Vega and A. Gonzalez-Ruiz ,J.Phys.A 26 (1994)6129 .

[8] M.T.Batchelor ,Reflection Equations and Surface Critical Phenomena, ANU preprint.

[9] Y.K.Zhou,Fusion Hierachies with Open boundaries and Exactly Solvable Models, ANU preprint MRR 079-95.

[10] H.Fan , B.Y.Hou ,K.J.Shi ,W.L.Yang and Z.X.Yang, J.Phys. A28 (1995) 3157.

[11] B.Y.Hou and Y.K.Zhou ,J.Phys.A 23(1990) 1147.

[12] L.Mezincescu and R.I.Nepomechie ,Int.J.Mod.Phys.A 6(1991)5231.